Math 261B Tues. Nov. 24

$\mathcal{O}_q(SL_2)$ = subalgebra of $\mathcal{U}_q(sl_2)^*$ spanned by matrix coefficients of the V^m

- $V^m \otimes V^n \cong V^{m+n} \oplus V^{m+n-2} \oplus \cdots \oplus V^{|m-n|}$

  $\Rightarrow$ it is a subalgebra

  $V^1 \otimes V^n \cong V^{n+1} \oplus V^{n-1}$

  $\Rightarrow$ matrix coef's of V^1 $\quad \begin{array}{ccc} & \bullet\, u & \circlearrowleft q \\ E \uparrow & \downarrow F & K \\ & \bullet\, v & \circlearrowleft q^{-1} \end{array} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

  a, b, c, d generate $\mathcal{O}_q(SL_2)$

- Coproduct comes from matrix multiplication

$$\Delta a_{ij} = \sum_k a_{ik} \otimes a_{kj}$$

- Product: a, b, c, d satisfy some relations.

$$V^1 \otimes V^1 \cong V^2 \otimes V^0$$

$$\left(\begin{array}{c|c} V^2 & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0\,0\,0 & V^0 \end{array}\right)$$

V^0 is the "trivial" rep.: its matrix entry is the counit $\varepsilon : U_q(sl_2) \to k$, i.e. 1 in $\mathcal{O}_q(SL_2)$

$$\begin{tikzcd}[column sep=small] & u\otimes u \arrow[dl, dash] \arrow[dr, dash] & \\ u\otimes v \arrow[dr, dash] & & v\otimes u \arrow[dl, dash] \\ & v\otimes v & \end{tikzcd} \cong \begin{tikzcd} u\otimes u \arrow[d, dash] \\ u\otimes v + q^{-1}\, v\otimes u \arrow[d, dash] \\ v\otimes v \end{tikzcd} \oplus \quad v\otimes u - q^{-1} u\otimes v$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{aligned} u &\mapsto ua + vc \\ v &\mapsto ub + vd \end{aligned} \qquad u\otimes u \mapsto (ua+vc)\otimes(ua+vc)$$

Suppress $\otimes$ signs

$$uu \mapsto (ua+vc)(ua+vc) = uu\, a^2 + uv\, ac + vu\, ca + vv\, c^2$$

$$= uu\, a^2 + (uv + q^{-1} vu)\,\frac{q\,ac + ca}{q+q^{-1}} + (vu - q^{-1} uv)\,\frac{q\,ca - ac}{q+q^{-1}} + vv\, c^2$$

$$\boxed{\begin{aligned} &uv\, X + vu\, Y \\ &= (uv + q^{-1} vu)\,\frac{qX+Y}{q+q^{-1}} \\ &\quad + (vu - q^{-1} uv)\,\frac{qY - X}{q+q^{-1}} \end{aligned}}$$

$$uv \mapsto (ua+vc)(ub+vd) = uu\,ab + uv\,ad + vu\,cb + v^2\,cd$$

$$vu \mapsto (ub+vd)(ua+vc) = uu\,ba + uv\,bc + vu\,da + v^2\,dc$$

$$uv + q^{-1}vu \mapsto uu\,(ab+q^{-1}ba) + uv\,(ad+q^{-1}bc) + vu\,(cb+q^{-1}da) + v^2\,(cd+q^{-1}dc)$$

$$= uu\,(ab+q^{-1}ba) + (uv+q^{-1}vu)\,\frac{q\,ad+bc+cb+q^{-1}da}{q+q^{-1}}$$

$$+ (vu-q^{-1}uv)\,\frac{q\,cb+da-ad-q^{-1}bc}{q+q^{-1}} + v^2(cd+q^{-1}dc$$

$$vu - q^{-1}uv \mapsto uu\,(ba-q^{-1}ab) + uv\,(bc-q^{-1}ad) + vu\,(da-q^{-1}cb) + v^2\,(dc-q^{-1}cd)$$

$$= uu\,(ba-q^{-1}ab) + (uv+q^{-1}vu)\,\frac{q\,bc-ad+da-q^{-1}cb}{q+q^{-1}}$$

$$+ (vu-q^{-1}uv)\,\frac{q\,da-cb-bc+q^{-1}ad}{q+q^{-1}} + v^2\,(dc-q^{-1}cd)$$

$$vv \mapsto (ub+vd)(ub+vd) = uu\,b^2 + uv\,bd + vu\,db + v^2\,d^2$$

$$= uu\,b^2 + (uv+q^{-1}vu)\,\frac{q\,bd+db}{q+q^{-1}} + (vu-q^{-1}uv)\,\frac{q\,db-bd}{q+q^{-1}} + v^2 d^2$$

Relations: The following are 0 in $\mathcal{O}_q(SL_2)$:

$ac - qca$
$bd - qdb$
$ab - qba$
$cd - qdc$

$$q\,cb + da - ad - q^{-1}bc$$
$$q\,bc - ad + da - q^{-1}cb$$
$$(q+q^{-1})cb - (q+q^{-1})bc$$
$$\Rightarrow cb - bc$$

$$ad - da = (q-q^{-1})bc$$

Also $q\,da - cb - bc + q^{-1}ad = q + q^{-1}$ $\Rightarrow$ $q\,da + q^{-1}ad - 2bc = q + q^{-1}$

$$da = ad + (q^{-1} - q)bc$$
$$q\,da = q\,ad + (1-q^2)bc$$
$$(q+q^{-1})ad - (1+q^2)bc = q + q^{-1}$$
$$ad - qbc = 1$$

$\mathcal{O}_q(SL_2) = k\langle a,b,c,d\rangle$ /

$\begin{matrix} a & b \\ c & d \end{matrix}$

$$\left(\begin{aligned} ac &= qca \\ ab &= qba \\ bd &= qdb \\ cd &= qdc \\ bc &= cb \\ [a,d] &= (q-q^{-1})bc \\ ad - qbc &= 1 \\ da - q^{-1}bc &= 1 \end{aligned} \right\} \right)$$

Nice q-analog of $\mathcal{O}(SL_2)$

General G Given X $\alpha_1, \dots, \alpha_r \in X$, $\alpha_1^\vee, \dots, \alpha_r^\vee \in X^*$

$\rightarrow$ Cartan matrix $A_{ij} = \langle \alpha_j^\vee, \alpha_i \rangle$

Choose symmetrizing numbers: $d_i \in \mathbb{Z}_+$ such that

$$d_j \langle \alpha_j^\vee, \alpha_i \rangle = d_i \langle \alpha_i^\vee, \alpha_j \rangle$$

i.e. $A \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_r \end{pmatrix}$ is symmetric. (It's possible!)

$k = \mathbb{Q}(q)$ ($\mathbb{Z}[q^{\pm 1}]$ later)

Def $U_q(\mathfrak{g})$: gen. by $\cdot\ \mathcal{O}_q(T^\vee) = k \cdot X^*$

$= k \cdot \{ K^\beta \mid \beta \in X^* \}$

$E_1, \dots, E_r$, $F_1, \dots, F_r$

Relations:

$$K^\beta E_i K^{-\beta} = q^{\langle \beta, \alpha_i \rangle}$$

$$K^\beta F_i K^{-\beta} = q^{-\langle \beta, \alpha_i \rangle}$$

$$[E_i, F_j] = 0 \text{ if } i \neq j$$

$K^\beta E_i = q^{\langle \beta, \alpha_i \rangle} E_i K^\beta$

$V = \mathcal{O}_q(T)$ comodule:

$V = \bigoplus_{\lambda \in X} V_\lambda$

$K^\beta v = q^{\langle \beta, \lambda \rangle} v$

$v \in V_\lambda$

$$[E_i, F_i] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$V_\lambda \underset{E_i}{\overset{F_i}{\rightleftarrows}} V_{\lambda+\alpha_i}$$

A, D, E have symmetric Cartan matrix.

where $K_i = K^{d_i \alpha_i^\vee}$ $\qquad q_i = q^{d_i}$

$E_i, F_i, K_i^{\pm 1}$ generate a copy of $U_{q_i}(sl_2)$

And <u>quantum Serre relations</u> (in E's) (in F's)

$$U_q(\mathfrak{g}) \underset{\substack{\uparrow \\ \text{via multiplication}}}{=} \underset{\langle F_i \rangle}{\underset{\shortparallel}{U_q(\mathfrak{n}_-)}} \otimes \underset{\text{``}U_q(\mathfrak{t})\text{''}}{\underset{\shortparallel}{\mathcal{O}_q(T^\vee)}} \otimes \underset{\langle E_i \rangle}{\underset{\shortparallel}{U_q(\mathfrak{n}_+)}}$$

- Coproduct:

$$\Delta K^\beta = K^\beta \otimes K^\beta$$

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i$$

$$\Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i$$

Check that $[E_i, F_j] = 0$ is consistent with this Δ:

$$[\Delta E_i, \Delta F_j] = [E_i \otimes 1 + K_i \otimes E_i, F_j \otimes K_j^{-1} + 1 \otimes F_j]$$

$$= [K_i \otimes E_i, F_j \otimes K_j^{-1}] \overset{?}{=} 0$$

$$K_i \otimes E_i \cdot F_j \otimes K_j^{-1} = K_i F_j \otimes E_i K_j^{-1} = q^{-d_i \langle \alpha_i^\vee, \alpha_j \rangle} F_j K_i \otimes E_i K_j^{-1}$$

$$F_j \otimes K_j^{-1} \cdot K_i \otimes E_i = F_j K_i \otimes K_j^{-1} E_i = q^{-d_j \langle \alpha_j^\vee, \alpha_i \rangle} F_j K_i \otimes E_i K_j^{-1}$$